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COMPRESSION WAVES IN A FLUID WITH GAS BUBBLES*

A. YU. BELYAYEV

For a fluid with gas bubbles, the wave equation with a fourth-order mixed derivative with respect to time and the coordinate is proved. The proof is based on the Bloch-Floquet form whose parameters are found by solving a problem in a cell of periodicity. The identity of the dispersion relations is used. Dispersion and wave-damping effects were studied in the case of acoustic oscillations in a fluid with gas bubbles in /1-3/. On the basis of heuristic arguments an equation was proposed in /2, 3/ which describes the propagation of acoustic waves in such a medium. Neglecting surface tension and compression of the fluid, this equation can be reduced to the form

$$3cp_{tt} = 3a^2\rho\rho_1^{-1}\Delta p + R^2\Delta p_{tt}$$

where ρ_1 and ρ are the fluid and gas densities, a is the velocity of sound in the gas, R is the radius of the bubble, c is the volume concentration of the gas, p is the pressure in the mixture, and Δ is the Laplace operator. The presence of the last term on the right-hand side leads, see /2, 3/, to dispersion of the waves, or for waves with frequencies that exceed $aR^{-1}(3\rho\rho_1^{-1})^{1/2}$, to exponential damping. Our present aim is to deduce the equation from the model that describes the microstructure of the suspension, by using the theory of averaging of equations with rapidly oscillating coefficients /4/. This approach, as distinct from the phenomenological approach, enables a connection to be found between the microscopic and macroscopic parameters of the motion of the suspension, and enables the error of the averaged equation to be estimated. When obtaining the model of the microstructure, the arrangement of the bubbles is assumed to be symmetric in the equilibrium state, so that the averaging problem can be reduced to a problem in a cell of periodicity. The solution of the cell problem and the coefficients of the averaged equation are expressed in terms of special harmonic functions which represent an extension of the Weierstrass functions to the case of dynamics.

1. Formulation of the problem. The acoustic waves in a fluid with gas bubbles are given by the equation

$$a^{-2}(x)\rho^{-1}(x)\varphi_{tt} = \nabla_\alpha\rho^{-1}(x)\nabla^\alpha\varphi \tag{1.1}$$

where $\varphi(x, t)$ is the potential of the pulses or pressure in the medium, $\rho(x)$ and $a(x)$ are the density and velocity of sound in the equilibrium state, in the neighbourhood of which the equations of gas dynamics are linearized: the indices α run over the values 1, 2, 3, and summation is performed with respect to them. In a uniform fluid with homogeneous gas bubbles, the functions $\rho(x)$ and $a(x)$ are piecewise-constant and take the values ρ_1, a_1, and ρ, a in the domains occupied by the fluid and gas respectively. On the boundaries between the fluid and gas, if surface tension is neglected, we have to assume that the function φ and the normal component of the vector $\rho^{-1}(x)\nabla\varphi$ are continuous.

We assume that, in the equilibrium state, all the bubbles are spherical, of the same radius R, and are arranged periodically in space, i.e., their centres are at the nodes of a periodic lattice with generating vectors τ_α $(\alpha = 1, 2, 3)$. We locate the origin at a bubble centre. The centres of the other bubbles are then at the points $x_m = m^\alpha\tau_\alpha$, where m^α are integers. The vectors τ_α are the periods of the coefficients of Eq.(1.1).

We consider the case when the characteristic scale of variation of the initial data for Eq.(1.1) is much greater than the distance between adjacent bubbles. In this case, Eq.(1.1) can be averaged. This property can be stated in terms of the theory of Bloch-Floquet waves /5, 6/, i.e., of solutions of Eq.(1.1) of the type

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$$\varphi(x, t) = \Phi(x) \exp i(kx - \omega t) \tag{1.2}$$

where ω and k are the frequency and wave vector of the Bloch-Floquest wave, which are connected by the dispersion relation, and $\Phi(x)$ is a periodic function with the same cell of periodicity as the coefficients of Eq.(1.1).

To find the frequency ω and the Bloch function Φ with a given wave number k, we subsitute (1.2) into Eq.(1.1) and obtain a selfadjoint spectral problem, which can be given the variational statement /6/

$$\delta J(\chi) = 0 \tag{1.3}$$
$$J(\chi) = \int_B \rho^{-1}(x) |\nabla\chi|^2 d^3x \left(\int_B a^{-2}(x) \rho^{-1}(x) |\chi|^2 d^3x\right)^{-1}$$

where B is the cell of periodicity of the coefficients of the equation. The functional J is defined on the set of complex-valued functions $\chi(x)$, which satisfy the multiplicativeness conditions

$$\chi(x + \tau_\alpha) = \chi(x) \exp ik\tau_\alpha \tag{1.4}$$

These conditions play the role of boundary conditions for the Euler equations of the variational problem (1.3).

The stationary values of the functional J are equal to the squares of the frequencies ω corresponding to the wave number k, i.e., they define the dispersion relation, while the stationary points $\chi(x)$ are connected with the periodic Bloch function $\Phi(x)$ by the relation

$$\chi(x) = \Phi(x) \exp ikx$$

For each real wave vector k, the functional J has a denumerable set of stationary values, and the dispersion relation splits into a denumerable set of dispersion branches $\omega^2 = \omega_s^2(k)$ ($s = 0, 1, 2, \ldots$). The functions ω_s depend on k evenly and periodically with periods ν_β, which form a basis in the space of wave vectors which is related to the basis of vectors τ_α:

$$\nu_\beta \tau_\alpha = 2\pi\delta_{\alpha\beta}$$

The lowest dispersion branch $\omega_0(k)$ passes through the point $\omega = 0$, $k = 0$. Its behaviour in the neighbourhood of this point is of direct concern for averaging theory.

The following theorem holds. Let $L(\omega, k)$ be even with respect to ω and complete with respect to k, such that $L(\omega_0(k), k) = O(|k|^N)$, $k \to 0$. Let $\varphi^\varepsilon(x, t)$ be the solution of the Cauchy problem for Eq.(1.1), in which the coefficients $\rho(x)$ and $a(x)$ are replaced by $\rho(x/\varepsilon)$ and $a(x/\varepsilon)$, where ε is a small parameter, and the initial data are independent of ε. Then,

$$L(i\varepsilon\partial/\partial t, i\varepsilon\nabla)\varphi^\varepsilon = O(\varepsilon^N)$$

as $\varepsilon \to 0$, where $O(\varepsilon^N)$ is understood in the sense of the convergence of distributions. This means that, if we take the integral convolution of φ^ε with any finite smooth function $h(x, t)$ and act on the convolution with the operator $\varepsilon^{-N}L$, we obtain a quantity which is uniformly bounded with respect to x and t as $\varepsilon \to 0$.

This theorem is a consequence of the results of /4/. It enables us to replace Eq.(1.1) with variable coefficients by equations with any degree of approximation, whose coefficients are constant. Approximation of the lowest branch of the dispersion relation by a second-degree polynomial $L(\omega, k) = \omega^2 - a^{\alpha\beta}k_\alpha k_\beta$ corresponds to the value $N = 4$ and leads to a second-order averaged equation, i.e., the equation of acoustics with constant coefficients. Higher-order approximations lead to equations which describe the wave dispersion. The averaging problem thus reduces to solving the cell problem which consists in minimizing the functional J.

2. Simplification of the cell problem. For actual fluids and gases, we usually have the inequalities

$$\rho \ll \rho_1, \quad a^2\rho \ll a_1^2\rho_1$$

which allow the problem of finding the lowest branch of the dispersion relation to be simplified. Physically, the simplifications amount to regarding the fluid as incompressible in the principal approximation with respect to these parameters, and the pressure in each bubble as constant with respect to the space coordinates. Under these assumptions, the functional J can be written as

$$J = \frac{3a^2\rho}{4\pi R^3\rho_1 |\chi_0|^2} \int_{B_1} |\nabla\chi|^2 d^3x$$

where B_1 is the part of the cell B occupied by fluid, and χ_0 is the constant value of the function $\chi(x)$ in the bubble located in B. The quantity χ_0 only affects the normalization of the eigenfunctions $\chi(x)$ and henceforth is put equal to R^{-1}.

These simplifications were justified in /7/. It must be said that they are only permissible for evaluating the lowest branch of the dispersion relation. For the higher frequencies, both the fluid compressibility and the gas non-uniformity in the bubbles are important. Moreover, the fluid compressibility cannot be ignored at very low gas concentrations. The latter must be much greater than $a^2\rho a_1^{-2}\rho_1^{-1}$, though even then it is well below unity.

As distinct from the initial problem, the variational problem (1.3) with the simplified form of the functional J, has a unique stationary point, and amounts to finding the function $\chi(x)$, complex-valued and harmonic in the domain occupied by the fluid, which satisfies the multiplicativeness condition (1.4) and takes the value $\chi_0 = R^{-1}$ on the boundary with a bubble.

3. Construction of the multiplicative Green's function. We consider the following method of solving the variational problem (1.3). Let $\Gamma^k(x)$ be a function which satisfies the multiplicativeness conditions and is a solution of Poisson's equation

$$\Delta\Gamma^k(x) = -4\pi\Sigma\delta(x - x_m)\exp ikx_m$$

where the summation is over the centres of the bubbles x_m. The existence of such a function is proved below. The solution of problem (1.3) is sought as the series

$$\chi(x) = C\Gamma^k(x) + \sum_{l=1}^{\infty} R^l C^{\alpha_1\ldots\alpha_l}\Gamma^k_{\alpha_1\ldots\alpha_l}(x) \tag{3.1}$$

where C, $C^{\alpha_1\ldots\alpha_l}$ are constants, and the subscripts $\alpha_1\ldots\alpha_l$ in the function $\Gamma^k(x)$ indicate that it is differentiated with respect to the coordinates and that summation is performed over them. The fact that the function $\chi(x)$ thus constructed is harmonic and multiplicative is obvious, if the series (3.1) converges sufficiently rapidly, while the condition on the boundary with the bubbles can be satisfied by a suitable choice of the constants C, $C^{\alpha_1\ldots\alpha_l}$.

This method of solution was used in /8-10/ when averaging periodic structures in statics, and periodic harmonic functions were constructed, in terms of which the solution of the cell problem is expressed. Their generalization for problems of dynamics are the multiplicative harmonic functions $\Gamma^k(x)$.

The existence of the function $\Gamma^k(x)$ can be proved by writing it as one of the Fourier series

$$\Gamma^k(x) = \Sigma\frac{1}{|x - x_m|}\exp ikx_m$$

$$\Gamma^k(x) = \frac{4\pi}{|B|}\Sigma\frac{1}{|k - k_m|^2}\exp i(k - k_m)x$$

where $|B|$ is the volume of the cell B. In the first series, the summation is over the all bubble centres x_m, and in the second, over the nodes of the related lattice $k_m = m^\alpha v_\alpha$. Both series are conditionally convergent; this is proved in /8/ with the aid of the following well-known device: they are differentiated term by term with respect to x or k a sufficient number of times so that the series become absolutely convergent, then $\Gamma^k(x)$ is restored by integration.

It can be shown by direct substitution that the function constructed by means of these series satisfies Poisson's equation and condition (1.4). The convergence of both forms is destroyed when $x = x_m$ and $k = k_m$. For these values of the arguments, $\Gamma^k(x)$ has singularities.

We shall require later some properties of the function $Q^k(x) = \Gamma^k(x) - |x|^{-1}$. This function is harmonic with respect to x and periodic in k with periods v_α. The quantity $Q^k(0)$ is real, depends evenly on k, and behaves asymptotically as $k \to 0$ as follows:

$$Q^k(0) = \frac{4\pi}{|B||k|^2} - q\left(\frac{4\pi}{3|B|}\right)^{1/3} + O(|k|^2)$$

The constant q depends on the geometry of the lattice and can be found numerically. For a cubic, face-centred, and body-centred lattice, $q \approx 1.8$.

4. Solution of the cell problem. The stationary point of the variational problem (1.3) is sought as the series (3.1). The choice of the constants C, $C^{\alpha_1\ldots\alpha_l}$ is not unique, since the functions $\Gamma^k_{\alpha_1\ldots\alpha_l}(x)$ are linearly dependent: they are symmetric with respect to permutations of the indices and their convolution with respect to any pair of indices is zero. To avoid this arbitrariness, we have to subject the constants $C^{\alpha_1\ldots\alpha_l}$ to the same linear relations. In this case the constants can be uniquely defined by expanding the left- and right-hand sides of Eq.(3.1) in spherical harmonics on the sphere $|x| = R$. After multiplying (3.1) by $\nabla_{\beta_1}\ldots\nabla_{\beta_s}|x|^{-1}$ and integrating over the sphere, we obtain a system of linear algebraic equations of infinite order for finding the constants C, $C^{\alpha_1\ldots\alpha_l}$. This system can be reduced to the form

$$C + R\sum_{l=0}^{\infty} R^l C^{\alpha_1\ldots\alpha_l} Q^k_{\alpha_1\ldots\alpha_l}(0) = 1 \tag{4.1}$$

$$C^{\beta_1\ldots\beta_s} + \frac{(-2)^s}{(2s)!} R^{s+1} \sum_{l=0}^{\infty} R^l C^{\alpha_1\ldots\alpha_l} Q^k_{\alpha_1\ldots\alpha_l\beta_1\ldots\beta_s}(0) = 0 \tag{4.2}$$

where $s = 1, 2, 3, \ldots$, and the Greek subscripts of $Q^k(x)$ indicate its differentiation with respect to the coordinates.

When $R = 0$ the system is easily solved. When $R > 0$ the following method of solution may be proposed. We fix the value of the constant C and solve system (4.2) by successive approximations for $C^{\alpha_1\ldots\alpha_l}$, $l > 0$. As the zero approximation we put $C^{\alpha_1\ldots\alpha_l} = 0$ and substitute these values into (4.2) under the summation sign. Hence we find the first approximation

$$C^{\beta_1\ldots\beta_s} = -[(-2)^s/(2s)!]R^{s+1} C Q^k_{\beta_1\ldots\beta_s}(0)$$

We then substitute into (4.2) under the summation sign the values of the constants found at the previous step, and obtain the next approximation, etc. All the approximations, starting with the second, are infinite series in powers of R. Our procedure leads to a power series expansion in the bubble radii of the solution of system (4.2). The convergence of the method for sufficiently small R can be proved by the principle of contracted mappings.

The constant C is found from Eq.(4.1), into which the solution of system (4.2) is substituted. If we confine ourselves to the zero approximation, we obtain the expression

$$C = (1 + RQ^k(0))^{-1} \tag{4.3}$$

Its relative error as $R \to 0$ is $O(R^4)$ for each fixed $k \neq k_m$ and $O(R^3)$ uniformly with respect to k.

5. Analysis of the dispersion relation.

Knowing the stationary point $\chi(x)$ of the functional J, we can calculate its stationary value. After integration by parts and using Eq.(3.1), the expression for the stationary value reduces to the form $\omega^2 = \Omega^2 C$, where $\Omega^2 = 3a^2\rho R^{-2}\rho_1^{-1}$, and the constant C is found from the system (4.2), (4.1), and depends on the wave vector k via the coefficients of this system. Given sufficiently small gas concentrations, the asymptotic relation (4.3) can be used and the following approximate form of the dispersion relation obtained:

$$\omega^2 = \Omega^2(1 + RQ^k(0))^{-1}$$

The quantity $Q^k(0)$ has a singularity for $k = k_m$, so that the frequency ω vanishes for $k = k_m$. For sufficiently small R, the frequency has an upper limit set by a quantity of order Ω.

For frequencies exceeding the maximum possible value, there are no Bloch-Floquest waves which are either increasing or damped at infinity. This means that disturbances with these frequencies cannot propagate in the medium without exponentially rapid damping. An anomalous increase in the damping decrement for waves with frequencies of order Ω has been observed experimentally /1/.

To obtain the averaged equation of any order of approximation, we must, in accordance with our theorem, approximate the dispersion relation by a polynomial in the neighbourhood of the point $\omega = 0$, $k = 0$. We shall confine ourselves to the averaged equation of fourth order. Using the expansion of the function $Q^k(0)$ at the singular point, we can write the required fourth-degree polynomial as

$$L(\omega, k) = 3c\omega^2 - \Omega^2 R^2 |k|^2 + R^2(1 - c^{1/3}q)|k|^2\omega^2$$

The relative error with which the coefficients of this polynomial are evaluated is $O(c)$ as $c \to 0$. If we neglect $c^{1/3}q$ in the last term by comparison with unity, the averaged equation

$$L(i\partial/\partial t, i\nabla)\varphi(x, t) = 0$$

is just the same as that obtained by phenomenological theory. We can refine the operator L, both for larger values of the gas concentration, and for larger values of the wave vector and frequency. The refining terms depend, however, on the type of periodic lattice of the bubble centres, and thus have no direct relation to an actual suspension, in which the bubbles are arranged chaotically.

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ASYMPTOTIC ANALYSIS OF CONVECTIVE DIFFUSION IN PROBLEMS WITH A DISCONTINUITY IN THE CATALYTIC PROPERTIES OF THE SURFACE AROUND WHICH THE FLOW TAKES PLACE*

V.G. KRUPA and G.A. TIRSKII

The stationary concentration distribution in the flow round bodies with a discontinuity in their surface catalytic properties is investigated. An asymptotic analysis of this problem is carried out on the basis of the Navier-Stokes equations when $\mathrm{Re}\to\infty$ in the neighbourhood of the point of discontinuity in the catalytic properties, and the corresponding boundary value problems for the leading terms in the expansion of the required functions are formulated. Two spatial problems are solved in which account is taken of the transverse diffusion during the circumfluence of a planar surface with a rectangular insert endowed with different catalytic properties. Cases are considered when the diffusion flux of recombining particles changes in a stepwise manner on-passing over the surface of the insert and when the main surface is non-catalytic but the insert is ideally catalytic.

1. Let us consider the stationary flow of a bindary mixture of a chemically reacting compressible gas along the surface of a disc and let this disc have a discontinuity in its surface catalytic properties at a distance $x_0 = O(1)$ from the origin. We assume that $\varepsilon = \mathrm{Re}^{-1/2}\to 0$, $\mathrm{Re} = \rho_\infty V_\infty x_0/\mu_\infty$ (quantities with the subscript ∞ correspond to values of the parameters in the approach stream). In a rectangular system of Cartesian coordinates x, y (see /1/, for example), the Navier-Stokes equations in the dimensionless variables

$$x^* = \frac{x-x_0}{x_0},\quad y^* = \frac{y}{x_0},\quad \rho^* = \frac{\rho}{\rho_\infty},\quad \mathbf{v}^* = \frac{\mathbf{v}}{V_\infty},\quad k^* = \frac{k}{V_\infty}$$
$$T^* = \frac{T}{T_\infty},\quad T_\infty = \frac{V_\infty{}^2}{c_{p\infty}},\quad p^* = \frac{p}{\rho_\infty V_\infty{}^2},\quad h^* = \frac{h}{c_{p\infty}T_\infty},$$
$$w^{\cdot *} = \frac{w^{\cdot} x_0}{\rho_\infty V_\infty}$$

take the form (we shall omit the asterisks above the dimensionless quantities)

$$\nabla\cdot\rho\mathbf{v} = 0,\ \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \varepsilon^2\nabla\cdot\boldsymbol{\tau} \tag{1.1}$$
$$\rho\mathbf{v}\cdot\nabla c = \varepsilon^2\,\nabla(\mu\,\mathrm{Sc}^{-1}\nabla c) + w^{\cdot}$$

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